Approximation-Based Adaptive Tracking Control of Pure-Feedback Nonlinear Systems with Multiple Unknown Time-Varying Delays

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***Abstract*— This paper presents adaptive neural tracking control for a class of non-affine pure-feedback systems with multiple unknown state time-varying delays. To overcome the design difficulty from non-affine structure of pure-feedback system, mean value theorem is exploited to deduce affine appearance of state variables x_i as virtual controls α_i, and of the actual control u. The separation technique is introduced to decompose unknown functions of all time-varying delayed states into a series of continuous functions of each delayed state. The novel Lyapunov–Krasovskii functionals are employed to compensate for the unknown functions of current delayed state, which is effectively free from any restriction on unknown time-delay functions and overcomes the circular construction of controller caused by the neural approximation of a function of u and $\dot{u}$. Novel continuous functions are introduced to overcome the design difficulty deduced from the use of one adaptive parameter. To achieve uniformly ultimate boundedness of all the signals in the closed-loop system and tracking performance, control gains are effectively modified as a dynamic form with a class of even function, which makes stability analysis be carried out at the present of multiple time-varying delays. Simulation studies are provided to demonstrate the effectiveness of the proposed scheme.**

***Index Terms*— Adaptive control, backstepping, neural network, nonlinear time-delay systems, pure-feedback systems.**

I. Introduction

In recent years, the systematic backstepping technique has become a powerful method for controlling nonlinear systems with a triangular structure. Adaptive backstepping design was first proposed in [1] to obtain global stability for parametric strict-feedback systems with overparameterization, and the overparameterization was overcome by the introduction of tuning functions in [2]. Adaptive backstepping design was further developed for a class of uncertain strict-feedback nonlinear systems with known or unknown constant input gains (see [3]–[5]). By combining novel Lyapunov functions with neural networks or fuzzy logic systems, adaptive backstepping design was extensively used to control strict-feedback nonlinear systems with known or unknown function input gains (see [6]–[10]). Pure-feedback nonlinear systems that have a more representative form than strict-feedback systems have no affine appearance of state variables to be used as virtual controls and the actual control. This makes the control design of pure-feedback nonlinear systems difficult and challenging. In [11], a class of parametric pure-feedback systems with a triangular structure was studied by adaptive control. In [12], stable adaptive neural network control was proven rigorously for general nonlinear systems. Subsequently, some results appeared on pure-feedback nonlinear systems, e.g., [13]–[16]. Adaptive neural control was presented for a class of uncertain pure-feedback nonlinear systems with a control variable or virtual one in affine form [13], [15], [16]. Using the input-to-state stability analysis and small-gain theorem, adaptive neural control was proposed for a class of non-affine pure-feedback nonlinear systems [14].

Stability analysis and robust control for nonlinear time-delay systems have attracted considerable attention due to the great challenge in academic research and demands in industrial applications. Time delays are often encountered in many dynamic systems such as rolling mill systems, biological systems, metallurgical processing systems, network systems, and so on [17], [18]. Lyapunov–Krasovskii functionals [19] and Lyapunov–Razumikhin functions [20] are the two main tools in controlling nonlinear time-delay systems. By combining Lyapunov–Razumikhin functionals and backstepping, adaptive stabilizing control schemes were presented in [21] and [22] for a class of strict-feedback nonlinear time-delay systems with known control input constraints. By Lyapunov–Krasovskii functionals, the tracking control problem was solved in [23] for a class of nonlinear systems in a Brunovsky form. By Lyapunov–Krasovskii functionals and backstepping, the authors in [24] and [25] solved the tracking problem for a class of strict-feedback nonlinear time-delay systems

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with parameterized uncertainties. For a class of strict-feedback nonlinear time-delay systems with unknown virtual control coefficients and nonlinear time-delay terms, practical adaptive neural tracking controllers were successfully constructed in [26]–[28], which guarantees the boundedness of all the closed-loop signals and achieves tracking performance. Further development was shown in [29]–[32] for strict-feedback nonlinear time-delay systems. In [33] and [34], adaptive neural stabilizing control was proposed with the help of a novel Lyapunov–Krasovskii functional.

This paper presents a novel adaptive neural control to solve the tracking problem of a class of non-affine pure-feedback systems with multiple time-varying delays. The explicit controls are obtained using the mean value theorem. The novel Lyapunov–Krasovskii functionals are used to compensate for unknown functions with current delayed states. Radial basis function (RBF) neural networks are employed to approximate unknown packaged functions. The proposed control scheme guarantees the boundedness of all the signals in the closed-loop system. The main contributions of this paper are as follows:

1) the use of the separation technique [35] to decompose unknown functions of all time-varying delayed states into a series of positive continuous functions of each delayed state. This is not only free of any restriction on unknown time-delay functions, but also solves the design difficulty from each subsystem with all delayed states;
2) the use of quadratic-type Lyapunov functions to avoid the circular construction of controller for the considered pure-feedback system, when RBF neural networks are used to approximate the unknown nonlinear functions;
3) the use of norms of unknown neural weight vectors as the estimated parameters, which makes only an adaptation parameter to be tuned online. This significantly reduces the number of neural network input variables and alleviates the computational burden.

The rest of this paper is organized as follows. Section II gives the problem formulation and preliminaries. In Section III, adaptive neural control is proposed for a class of pure-feedback nonlinear systems with multiple unknown state time-varying delays using backstepping and appropriate Lyapunov–Krasovskii functionals, then the stability of the closed-loop system is proven rigorously. Simulation studies are performed to demonstrate the effectiveness of the proposed control scheme in Section IV. Finally, conclusions are included in Section V.

II. Problem Formulation and Preliminaries

Consider a class of pure-feedback nonlinear systems with unknown time-delay functions as follows:

$$\begin{cases} \dot{x}_i(t) = f_i(\bar{x}_i(t), x_{i+1}(t)) + h_i(\bar{x}_{n,\tau(t)}) \\ \dot{x}_n(t) = f_n(\bar{x}_n(t), u(t)) + h_n(\bar{x}_{n,\tau(t)}) \\ y(t) = x_1(t) \end{cases} \tag{1}$$

where $1 \le i \le n-1$, $\bar{x}_i(t) = [x_1(t), x_2(t), \ldots, x_i(t)]^T \in R^i$ with $i = 1, 2, \ldots, n$, $u(t) \in R$, and $y(t)$ are system state variables, system control input, and system output, respectively; $f_i(.)$ are unknown but smooth and non-affine functions, $h_i(.)$ with $h_i(0) = 0$ are unknown smooth nonlinear time-delay functions which are defined by $h_i(\bar{x}_{n,\tau(t)}) = h_i(x_1(t - \tau_1(t)), x_2(t - \tau_2(t)), \ldots, x_n(t - \tau_n(t)))$; $\tau_i(t)$ are unknown time-varying delays which satisfy $\tau_i(t) \le \tau$; and $\dot{\tau}_i(t) \le \tau_{\max} < 1$, $i = 1, 2, \ldots, n$, with τ and $\tau_{\max}$ being known constants. For $t \in [-\tau, 0]$, we have $\bar{x}_n(t) = \omega(t)$, with $\omega(t)$ being a known continuous initial state vector function. In what follows, the time variable t is omitted in the delay-free terms for brevity.

Remark 1: Non-affine structure in the considered pure-feedback nonlinear system (1) covers many dynamic systems such as rolling mill systems, biological systems, aircraft flight, and mechanical systems [36], [37]. It can be seen that the pure-feedback system (1) has no affine appearance of state variables x_i to be used as virtual controls α_i, and of the actual control u itself. The cascade and non-affine properties make it quite difficult to find the explicit virtual controls and the actual control using the backstepping design. Moreover, it can be shown in the system (1) that each state $x_i, i = 1, \ldots, n$ is assigned an independent time-varying delay $\tau_i(t)$, and the considered time-delay function $h_i(\bar{x}_{n,\tau(t)})$ contains not only the previous time-varying delay states $x_1(t - \tau_1(t)), \ldots, x_i(t - \tau_i(t))$, but also all the later delay states $x_{i+1}(t - \tau_{i+1}(t)), \ldots, x_n(t - \tau_n(t))$. These time-delay functions $h_i(\bar{x}_{n,\tau(t)})$ make controller design challenging and maybe cause the circular construction of controller caused by the neural approximation. Therefore, it is difficult and challenging to control the system (1).

Using the mean value theorem [38], we obtain the following explicit virtual control and actual control:

$$f_i(\bar{x}_i, x_{i+1}) = f_i(\bar{x}_i, x_{i0}) + g_{i\mu_i}(x_{i+1} - x_{i0}) \tag{2}$$

$$f_i(\bar{x}_n, u) = f_i(\bar{x}_n, x_{n0}) + g_{n\mu_n}(u - x_{n0}) \tag{3}$$

where $g_{i\mu_i} := g_i(\bar{x}_i, x_{\mu_i}) = \partial f_i(\bar{x}_i, x_{i+1})/\partial x_{i+1}|_{x_{i+1}=x_{\mu_i}}$ with $i = 1, 2, \ldots, n$, $x_{n+1} = u$, $x_{\mu_i} = \mu_i x_{i+1} + (1 - \mu_i)x_{i0}$, $0 < \mu_i < 1$, and x_{i0} are known at a given time value t_0. Then, substituting (2) and (3) into (1) yields

$$\begin{cases} \dot{x}_i = f_i(\bar{x}_i, x_{i0}) + g_{i\mu_i}(x_{i+1} - x_{i0}) + h_i(\bar{x}_{n,\tau(t)}) \\ \dot{x}_n = f_i(\bar{x}_n, x_{n0}) + g_{n\mu_n}(u - x_{n0}) + h_n(\bar{x}_{n,\tau(t)}) \\ y \ = x_1. \end{cases} \tag{4}$$

The control objective of this paper is to design adaptive neural tracking control such that the system output y follows a desired reference signal y_d, while all the signals in the closed-loop system remain uniformly ultimately bounded. To this end, define a vector function as $\bar{y}_{di} = [y_d, y_d^{(1)}, \ldots, y_d^{(i)}]^T$, $i = 1, 2, \ldots, n$, where $y_d^{(i)}$ is the ith time derivative of y_d.

Assumption 1: The desired trajectory vectors $\bar{y}_{di}$ are continuous and known, $\bar{y}_{di} \in \Omega_{di} \subset R^{i+1}$ with Ω_{di} being known compact sets, $i = 1, 2, \ldots, n$.

Assumption 2: The signs of nonlinear functions $g_i(.)$ are known, and there exists an unknown positive constant b such that $0 < b \le |g_i(.)| < \infty$, $\forall(\bar{x}_i, x_{i+1}) \in R^i \times R$. Without loss of generality, we further assume that $g_i(.) \ge b > 0$, $i = 1, 2, \ldots, n$.

In this paper, the following RBF neural network [39], [40] will be used to approximate any continuous function

$\varphi(Z)$: $R^q \to R$

$$\varphi_{nn}(Z) = W^T S(Z) \tag{5}$$

where $Z \in \Omega_Z \subset R^q$ is the input vector with q being the neural network input dimension, $W = [w_1, w_2, \ldots, w_l]^T \in R^l$ is the weight vector, $l > 1$ is the neural network node number, and $S(Z) = [s_1(Z), s_2(Z), \ldots, s_l(Z)]^T \in R^l$ is the basis function vector with $s_i(Z)$ chosen commonly as a Gaussian function in the following form:

$$s_i(Z) = \exp\left[\frac{-(Z-\xi_i)^T(Z-\xi_i)}{r^2}\right], \qquad i = 1, \ldots, l \tag{6}$$

where $\xi_i = [\xi_{i1}, \xi_{i2}, \ldots, \xi_{iq}]^T$ is the center of the receptive field and r is the width of the Gaussian function. As indicated in [40], the neural network (5) can approximate any continuous function $\varphi(Z)$ over a compact set $\Omega_Z \in R^q$ to any arbitrary accuracy ε as

$$\varphi(Z) = W^{*T} S(Z) + \delta(Z), \qquad \forall Z \in \Omega_Z \in R^q \tag{7}$$

where W^* is an ideal constant weight, and $\delta(Z)$ is the approximation error satisfying $|\delta(Z)| \le \varepsilon$. For the purpose of analysis, define W^* as an unknown ideal constant weight vector which is an artificial quantity. Typically, W^* is chosen as the value of W that minimizes $|\delta(Z)|$ for all $Z \in \Omega_Z$

$$W^* := \arg\min_{W \in R^l} \left\{\sup_{Z \in \Omega_Z} |\varphi(Z) - W^T S(Z)|\right\}.$$

For Gaussian RBF neural networks, the following lemma gives an upper bound on the norm of the basis function vector $S(Z)$, which is borrowed from [14].

Lemma 1 [14]: Consider the Gaussian RBF networks (5) and (6). Let $\rho = 1/2 \min_{i \ne j} \| \xi_i - \xi_j \|$, then an upper bound of $\|S(Z)\|$ is taken as $\| S(Z) \| \le \sum_{k=0}^{\infty} 3q(k+2)^{q-1} e^{-2\rho^2 k^2/r^2} := s$.

It has been proven in [41] and [14] that s is a limited value since the series $\{3q(k+2)^{q-1} e^{-2\rho^2 k^2/r^2},\ k = 0, 1, \ldots, \infty\}$ is convergent by the ratio test theorem [38]. Moreover, it is worth pointing out that the limited value s is independent of neural network inputs Z and neural network node numbers l.

Lemma 2 [35]: For any continuous function $h(\zeta_1, \ldots, \zeta_n) : R^{m_1} \times \cdots \times R^{m_n} \to R$, satisfying $h(0, \ldots, 0) = 0$, where $\zeta_j \in R^{m_j}$ $(j = 1, 2, \ldots, n, m_j > 0)$, there exist known positive smooth functions $\rho_j(\zeta_j) : R^{m_j} \to R$, $(j = 1, 2, \ldots, n)$ satisfying $\rho_j(0) = 0$ such that $|h(\zeta_1, \ldots, \zeta_n)| \le \sum_{j=1}^{n} \rho_j(\zeta_j)$.

Lemma 3 [29]: For $1 \le j \le n$, define the set $\Omega_{c_{z_j}}$ given by $\Omega_{c_{z_j}} := \{z_j \mid\mid z_j \mid < 0.2554\nu_j\}$. Then, for $z_j \notin \Omega_{c_{z_j}}$, the inequality $[1 - 16\tanh^2(z_j/\nu_j)] \le 0$ holds.

Lemma 4: Consider the dynamic system of the form $\dot{\chi}(t) = -a\chi(t) + cq(t)$, where a and c are positive constants and $q(t)$ is a positive function. Then, for any given bounded initial condition $\chi(t_0) \ge 0$, we have $\chi(t) \ge 0$ for $\forall t \ge t_0$.

Proof: For any given bounded initial condition $\chi(t_0)$, we obtain the solution to the equation $\dot{\chi}(t) = -a\chi(t) + cq(t)$ as

$$\chi(t) = e^{-a(t-t_0)}\chi(t_0) + \int_{t_0}^{t} e^{-a(t-\tau)} cq(\tau) d\tau. \tag{8}$$

Since c and $q(t)$ are positive, the integral term of (8) is also positive for $\forall t \ge t_0$. Therefore, (8) implies that under any given bounded initial condition $\chi(t_0) \ge 0$, $\chi(t) \ge 0$ for $\forall t \ge t_0$. This completes the proof of Lemma 4.

In this paper, to construct differentiable control laws, the following continuous even functions $\psi(x)$, $R \to R$, are introduced [29]:

$$\psi(x) = \frac{x^2 \cosh(x)}{1 + x^2}, \qquad \forall x \in R \tag{9}$$

which is continuous and monotonic, that is, for any given positive constant d, if $|x| > d$, then $\psi(x) > \psi(d)$.

III. Adaptive Neural Tracking Control

In this section, backstepping is used to develop an adaptive neural tracking control for pure-feedback nonlinear systems (1) with multiple time-varying delays. The backstepping design is based on the following coordinate transformation: $z_1 = x_1 - y_d$, $z_i = x_i - \alpha_{i-1}$, $i = 2, \ldots, n$, with α_i being virtual control laws. The actual control law u will be designed in the last step.

For the sake of clarity and convenience, denote $\|z\|$ as the Euclidean norm of vector z, i.e., $\|z\|^2 = z^T z$, define compact sets $\Omega^0_{Z_i}$ according to [26] and [27] as $\Omega^0_{Z_i} := \Omega_{Z_i} - \Omega_{c_{z_i}}$ with open set $\Omega_{c_{z_i}}$ in Lemma 3 and compact set Ω_{Z_i}, $i = 1, 2, \ldots, n$. We employ RBF neural networks to approximately package unknown functions $\hat{f}_i(Z_i)$ shown later as

$$\hat{f}_i(Z_i) = W_i^{*T} S_i(Z_i) + \delta_i(Z_i) \tag{10}$$

where $\delta_i(Z_i)$ is the approximation error and satisfies $|\delta_i(Z_i)| \le \varepsilon_i$, $S_i(Z_i)$ is the basis function vector of the RBF neural network, $Z_1 = [x_1, y_d, \dot{y}_d]^T \in \Omega^0_{Z_1} \subset R^3$, $Z_i = [x_1, x_2, \ldots, x_i, \hat{\theta}, \varsigma_{i-1}, \omega_{i-1}]^T \in \Omega^0_{Z_i} \subset R^{i+3}$, $2 \le i \le n$, are input vectors with ς_{i-1} and ω_{i-1} being defined later, W_i^* are unknown ideal constant weight vectors, construct the quadratic function V_{z_i} and the Lyapunov–Krasovskii functional candidate V_{P_i} in advance as

$$V_{z_i} = \frac{z_i^2}{2}, \tag{11}$$

$$V_{P_i} = \sum_{l=1}^{n} (n - l + 1) \int_{t-\tau_i(t)}^{t} \frac{\rho_{li}^2(x_i(\sigma))}{2(1 - \tau_{\max})} d\sigma \tag{12}$$

and construct the adaptive neural tracking controller for pure-feedback nonlinear time-delay system (1) as follows:

$$\alpha_i = -k_i z_i - \frac{\hat{\theta}}{2\eta_i^2} S_i^T(Z_i) S_i(Z_i) z_i + x_{i0}, \tag{13}$$

$$\dot{\hat{\theta}} = \sum_{i=1}^{n} \frac{\gamma}{2\eta_i^2} S_i^T(Z_i) S_i(Z_i) z_i^2 - \sigma\hat{\theta} \tag{14}$$

where $1 \le i \le n$, η_i, γ and σ are positive design parameters, $k_i = k_{i0} + k_{i1}$ with k_{i0} being positive design parameter, and $k_{i1} = k_{i2} + \epsilon_i \cosh(z_i)/1 + z_i^2 \sum_{l=1}^{n} (n - l + 1)\varsigma_i$, $\varsigma_i = \int_{t-\tau}^{t} \rho_{li}^2(x_i(\sigma))/2(1 - \tau_{\max}) d\sigma$, k_{i2} is a positive design parameter, $\hat{\theta}$ is the estimate of the unknown constant θ which is specified as $\theta = \max\{b^{-1} \left\| W_i^* \right\|^2,\ i = 1, 2, \ldots, n\}$ with b defined in Assumption 2. It should be pointed out that, when $i = n$, α_n is the actual control law u. In what follows, our control scheme is designed on the basis of compact sets $\Omega^0_{Z_i}$ for simplicity.

Remark 2: Note that (14) satisfies the conditions of Lemma 4. For any given initial condition $\hat{\theta}(t_0) \geq 0$, we thus have $\hat{\theta}(t) \geq 0$ for $t \geq t_0$. This property is very important and useful in our design. In fact, it is always reasonable to choose $\hat{\theta}(t_0) \geq 0$, since $\hat{\theta}(t)$ is the estimate of an unknown constant θ and its initial condition is an artificial value. Consequently, for clarity, we choose the initial condition $\hat{\theta}(t_0) = 0$ of the adaptation law $\hat{\theta}(t)$ in this paper.

Step 1: Noting $z_1 = x_1 - y_d$ and considering the transformed system (4), the error dynamic is

$$\dot{z}_1 = f_1(\bar{x}_1, x_{10}) + g_{1\mu_1}(x_2 - x_{10}) + h_1(\bar{x}_{n,\tau(t)}) - \dot{y}_d. \quad (15)$$

Choosing the quadratic function V_{z_1} in (11), its time derivative follows from (15):

$$\begin{aligned}\dot{V}_{z_1} &= z_1 f_1(\bar{x}_1, x_{10}) + z_1 g_{1\mu_1}(x_2 - x_{10}) \\ &\quad + z_1 h_1(\bar{x}_{n,\tau(t)}) - z_1 \dot{y}_d. \end{aligned} \quad (16)$$

Noting Lemma 2 and the definition of $h_1(\bar{x}_{n,\tau(t)})$, we have

$$z_1 h_1(\bar{x}_{n,\tau(t)}) \leq \frac{n z_1^2}{2} + \sum_{l=1}^{n} \frac{\rho_{1l}^2(x_l(t-\tau_l(t)))}{2}. \quad (17)$$

Substituting (17) into (16) yields

$$\begin{aligned}\dot{V}_{z_1} &\leq z_1 f_1(\bar{x}_1, x_{10}) + z_1 g_{1\mu_1}(x_2 - x_{10}) - z_1 \dot{y}_d \\ &\quad + \frac{n z_1^2}{2} + \sum_{l=1}^{n} \frac{\rho_{1l}^2(x_l(t-\tau_l(t)))}{2}. \end{aligned} \quad (18)$$

Now, take the Lyapunov–Krasovskii functional candidate V_{P_1} in (12) to compensate for the time-delay term in (18). The time derivative of V_{P_1} satisfies

$$\begin{aligned}\dot{V}_{P_1} &\leq \sum_{l=1}^{n} (n-l+1) \frac{\rho_{l1}^2(x_1)}{2(1-\tau_{\max})} \\ &\quad - \sum_{l=1}^{n} (n-l+1) \frac{\rho_{l1}^2(x_1(t-\tau_1(t)))}{2}. \end{aligned} \quad (19)$$

Define $V_1 = V_{z_1} + V_{P_1}$. It can be verified by (18) and (19) that

$$\begin{aligned}\dot{V}_1 &\leq z_1 \Bigg(f_1(\bar{x}_1, x_{10}) + \frac{n z_1}{2} + \sum_{l=1}^{n} \frac{(n-l+1)\rho_{l1}^2(x_1)}{2 z_1 (1-\tau_{\max})} \\ &\quad - \dot{y}_d + g_{1\mu_1}(x_2 - x_{10}) \Bigg) + \sum_{l=1}^{n} \frac{\rho_{1l}^2(x_l(t-\tau_l(t)))}{2} \\ &\quad - \sum_{l=1}^{n} \frac{(n-l+1)\rho_{l1}^2(x_1(t-\tau_1(t)))}{2}. \end{aligned} \quad (20)$$

Note that it is not feasible to use RBF neural network to approximate $\sum_{l=1}^{n}(n-l+1)\rho_{l1}^2(x_1)/(2z_1(1-\tau_{\max}))$ in (20) since it is discontinuous at $z_1 = 0$. Therefore, hyperbolic tangent function $\tanh(z_1/\nu_1)$ has to be introduced to overcome the design difficulty from the term $\sum_{l=1}^{n}(n-l+1)\rho_{l1}^2(x_1)/(2z_1(1-\tau_{\max}))$ [29]. In this way, (20) becomes

$$\begin{aligned}\dot{V}_1 &\leq z_1 \hat{f}_1(Z_1) + g_{1\mu_1} z_1 (x_2 - x_{10}) \\ &\quad + \left(1 - 16\tanh^2\left(\frac{z_1}{\nu_1}\right)\right) U_1 + \sum_{l=1}^{n} \frac{\rho_{1l}^2(x_l(t-\tau_l(t)))}{2} \\ &\quad - \sum_{l=1}^{n} \frac{(n-l+1)\rho_{l1}^2(x_1(t-\tau_1(t)))}{2} \end{aligned} \quad (21)$$

where

$$\hat{f}_1(Z_1) = f_1(\bar{x}_1, x_{10}) + \frac{n z_1}{2} + \frac{16\tanh^2\left(\frac{z_1}{\nu_1}\right)}{z_1} U_1 - \dot{y}_d,$$

$$U_1 = \sum_{l=1}^{n} \frac{(n-l+1)\rho_{l1}^2(x_1)}{2(1-\tau_{\max})}.$$

Note that $\lim_{z_1 \to 0} 16\tanh^2(z_1/\nu_1)U_1/z_1$ exists, and it is feasible to use RBF neural network (10) to approximate the nonlinear function $\hat{f}_1(Z_1)$ on the compact set $\Omega^0_{Z_1}$. Then, (21) becomes

$$\begin{aligned}\dot{V}_1 &\leq z_1 (W_1^{*T} S_1(Z_1) + \delta_1(Z_1)) + g_{1\mu_1} z_1 (x_2 - x_{10}) \\ &\quad + \left(1 - 16\tanh^2\left(\frac{z_1}{\nu_1}\right)\right) U_1 + \sum_{l=1}^{n} \frac{\rho_{1l}^2(x_l(t-\tau_l(t)))}{2} \\ &\quad - \sum_{l=1}^{n} \frac{(n-l+1)\rho_{l1}^2(x_1(t-\tau_1(t)))}{2}. \end{aligned}$$

By

$$\begin{aligned} z_1 W_1^{*T} S_1(Z_1) &\leq \frac{b\theta}{2\eta_1^2} S_1^T(Z_1) S_1(Z_1) z_1^2 + \frac{\eta_1^2}{2}, \\ z_1 \delta_1(Z_1) &\leq b k_{10} z_1^2 + \frac{\varepsilon_1^2}{4 b k_{10}} \end{aligned} \quad (22)$$

and noting $z_2 = x_2 - \alpha_1$, it can be easily verified that

$$\begin{aligned}\dot{V}_1 &\leq \frac{b\theta}{2\eta_1^2} S_1^T(Z_1) S_1(Z_1) z_1^2 + b k_{10} z_1^2 \\ &\quad + g_{1\mu_1} z_1 \alpha_1 - g_{1\mu_1} z_1 x_{10} + g_{1\mu_1} z_1 z_2 + d_1 \\ &\quad + \left(1 - 16\tanh^2\left(\frac{z_1}{\nu_1}\right)\right) U_1 + \sum_{l=1}^{n} \frac{\rho_{1l}^2(x_l(t-\tau_l(t)))}{2} \\ &\quad - \sum_{l=1}^{n} \frac{(n-l+1)\rho_{l1}^2(x_1(t-\tau_1(t)))}{2} \end{aligned} \quad (23)$$

where k_{10} and η_1 are positive design parameters, and $d_1 = \eta_1^2/2 + \varepsilon_1^2/(4bk_{10})$.

From the virtual control (13) and Remark 2, we have

$$g_{1\mu_1} z_1 (\alpha_1 - x_{10}) \leq -b k_1 z_1^2 - \frac{b\hat{\theta}}{2\eta_1^2} S_1^T(Z_1) S_1(Z_1) z_1^2. \quad (24)$$

Denote $\tilde{\theta} = \theta - \hat{\theta}$. Substituting (24) into (23) yields

$$\dot{V}_1 \leq -bk_{11}z_1^2 + \frac{b\tilde{\theta}}{2\eta_1^2}S_1^T(Z_1)S_1(Z_1)z_1^2 + g_{1\mu_1}z_1z_2 + \left(1 - 16\tanh^2\left(\frac{z_1}{\nu_1}\right)\right)U_1 + \sum_{l=1}^{n}\frac{\rho_{1l}^2(x_l(t-\tau_l(t)))}{2} - \sum_{l=1}^{n}\frac{(n-l+1)\rho_{l1}^2(x_1(t-\tau_1(t)))}{2} + d_1. \quad (25)$$

Remark 3: It can be seen from (16) that design difficulties mainly come from two nonlinear uncertainties: the coupling term $g_{1\mu_1}$ and the unknown time-delay term $h_1(\bar{x}_{n,\tau(t)})$. The two terms cannot appear in the designed controller due to the uncertainty, and not be also directly approximated by the use of neural networks which may result in the circular construction of controller. To overcome these difficulties, the following efforts have been made: $g_{1\mu_1}$ is effectively dealt with in (24) by Lemma 4, and only its lower bound is used for analysis purpose in this paper. From (17), $h_1(\bar{x}_{n,\tau(t)})$ is decomposed into a series of continuous functions $\rho_{1l}^2(x_l(t-\tau_l(t)))$, $l = 1, 2, \ldots, n$ of each delayed state $x_l(t-\tau_l(t))$, the time-varying delay function $\rho_{11}^2(x_1(t-\tau_1(t)))$ with current delayed state is compensated for by a Lyapunov–Krasovskii functional V_{P_1} in (12), the other time-varying delay functions $\rho_{1l}^2(x_l(t-\tau_l(t)))$, $l = 2, \ldots, n$, will be compensated for step by step, thus the time-delay term in (25) can be completely canceled in the last step of backstepping. The method is effectively free from any assumption on unknown time-delay functions $h_i(\bar{x}_{n,\tau(t)})$ and overcomes the circular construction of controller.

Step 2: For $z_2 = x_2 - \alpha_1$, we have

$$\dot{z}_2 = f_2(\bar{x}_2, x_{20}) + g_{2\mu_2}(x_3 - x_{20}) + h_2(\bar{x}_{n,\tau(t)}) - \dot{\alpha}_1$$

where

$$\dot{\alpha}_1 = \frac{\partial\alpha_1}{\partial x_1}h_1(\bar{x}_{n,\tau(t)}) + \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} + D_1 \quad (26)$$

with $D_1 = (\partial\alpha_1/\partial x_1)(f_1(\bar{x}_1, x_{10}) + g_{1\mu_1}(x_2 - x_{10})) + \omega_1$, $\omega_1 = (\partial\alpha_1/\partial\bar{y}_{d1})\dot{\bar{y}}_{d1} + (\partial\alpha_1/\partial\varsigma_1)\rho_{l1}^2(x_1) - \rho_{l1}^2(x_1(t-\tau))/2(1-\tau_{\max})$.

Choosing the quadratic function V_{z_2} in (11) and using the following inequalities:

$$z_2h_2(\bar{x}_{n,\tau(t)}) \leq \frac{nz_2^2}{2} + \sum_{l=1}^{n}\frac{\rho_{2l}^2(x_l(t-\tau_l(t)))}{2}$$

$$z_2\frac{\partial\alpha_1}{\partial x_1}h_1(\bar{x}_{n,\tau(t)}) \leq \frac{nz_2^2}{2}\left(\frac{\partial\alpha_1}{\partial x_1}\right)^2 + \sum_{l=1}^{n}\frac{\rho_{1l}^2(x_l(t-\tau_l(t)))}{2}$$

we have

$$\dot{V}_{z_2} \leq z_2\left(f_2(\bar{x}_2, x_{20}) + g_{2\mu_2}(x_3 - x_{20}) + \frac{nz_2}{2} + \frac{nz_2}{2}\left(\frac{\partial\alpha_1}{\partial x_1}\right)^2 - D_1\right) - z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} + \sum_{l=1}^{n}\frac{\rho_{2l}^2(x_l(t-\tau_l(t)))}{2} + \sum_{l=1}^{n}\frac{\rho_{1l}^2(x_l(t-\tau_l(t)))}{2}. \quad (27)$$

Now, use the Lyapunov–Krasovskii functional candidate V_{P_2} in (12) and define $V_2 = V_1 + V_{P_2} + V_{z_2}$, and the time derivative of V_2 along (25) and (27) is

$$\dot{V}_2 \leq -bk_{11}z_1^2 + \frac{b\tilde{\theta}}{2\eta_1^2}S_1^T(Z_1)S_1(Z_1)z_1^2 + d_1 + \left(1 - 16\tanh^2\left(\frac{z_1}{\nu_1}\right)\right)U_1 + z_2g_{2\mu_2}(x_3 - x_{20}) + z_2(f_2(\bar{x}_2, x_{20}) + g_{1\mu_1}z_1 + \frac{nz_2}{2} + \frac{nz_2}{2}\left(\frac{\partial\alpha_1}{\partial x_1}\right)^2 - D_1 + \sum_{l=1}^{n}\frac{(n-l+1)\rho_{l2}^2(x_2)}{2z_2(1-\tau_{\max})}) - z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} + \sum_{j=1}^{2}\sum_{l=1}^{n}\left[(2-j+1)\frac{\rho_{jl}^2(x_l(t-\tau_l(t)))}{2} - (n-l+1)\frac{\rho_{lj}^2(x_j(t-\tau_j(t)))}{2}\right]. \quad (28)$$

From (28), it is not practical to use RBF neural networks to approximate the term $\sum_{l=1}^{n}(n-l+1)\rho_{l2}^2(x_2)/(2z_2(1-\tau_{\max}))$ which does not exist as $z_2 = 0$. Similar to Step 1, it can employ the hyperbolic tangent function $\tanh(z_2/\nu_2)$ to compensate the term. Furthermore, a continuous function $\upsilon_1(Z_2)$ is introduced to overcome the design difficulty from the term $(\partial\alpha_1/\partial\hat{\theta})\dot{\hat{\theta}}$. Therefore, we have

$$\dot{V}_2 \leq -bk_{11}z_1^2 + \frac{b\tilde{\theta}}{2\eta_1^2}S_1^T(Z_1)S_1(Z_1)z_1^2 + d_1 + \sum_{j=1}^{2}\left(1 - 16\tanh^2\left(\frac{z_j}{\nu_j}\right)\right)U_j + z_2g_{2\mu_2}(x_3 - x_{20}) + z_2\hat{f}_2(Z_2) + z_2\left(\upsilon_1(Z_2) - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}\right) + \sum_{j=1}^{2}\sum_{l=1}^{n}\left[(2-j+1)\frac{\rho_{jl}^2(x_l(t-\tau_l(t)))}{2} - (n-l+1)\frac{\rho_{lj}^2(x_j(t-\tau_j(t)))}{2}\right] \quad (29)$$

where

$$\hat{f}_2(Z_2) = f_2(\bar{x}_2, x_{20}) + g_{1\mu_1}z_1 + \frac{nz_2}{2} + \frac{nz_2}{2}\left(\frac{\partial\alpha_1}{\partial x_1}\right)^2 - D_1 - \upsilon_1(Z_2) + \frac{16\tanh^2\left(\frac{z_2}{\nu_2}\right)}{z_2}U_2 \quad (30)$$

with $U_2 = \sum_{l=1}^{n}(n-l+1)\rho_{l2}^2(x_2)/2(1-\tau_{\max})$.

Next, using RBF neural networks (10) to approximate $\hat{f}_2(Z_2)$ on the compact set $\Omega_{Z_2}^0$ and, employing the similar method as (22), we obtain

$$\dot{V}_2 \le -bk_{11}z_1^2 + \frac{b\tilde{\theta}}{2\eta_1^2}S_1^T(Z_1)S_1(Z_1)z_1^2 + \sum_{j=1}^{2} d_j + \sum_{j=1}^{2}\left(1 - 16\tanh^2\left(\frac{z_j}{\nu_j}\right)\right)U_j + z_2 g_{2\mu_2} z_3 + z_2 g_{2\mu_2}\alpha_2 - z_2 g_{2\mu_2} x_{20} + \frac{b\theta}{2\eta_2^2}S_2^T(Z_2)S_2(Z_2)z_2^2 + bk_{20}z_2^2 + z_2\left(\upsilon_1(Z_2) - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}\right) + \sum_{j=1}^{2}\sum_{l=1}^{n}\left[(2-j+1)\frac{\rho_{jl}^2(x_l(t-\tau_l(t)))}{2} - (n-l+1)\frac{\rho_{lj}^2(x_j(t-\tau_j(t)))}{2}\right]$$

where $d_j = \eta_j^2/2 + \varepsilon_j^2/(4bk_{j0})$.

Considering the virtual control (13) and adaptation law (14), and similar to (24), we have

$$\dot{V}_2 \le \sum_{j=1}^{2}\left(-bk_{j1}z_j^2 + \frac{b\tilde{\theta}}{2\eta_j^2}S_j^T(Z_j)S_j(Z_j)z_j^2 + d_j\right) + \sum_{j=1}^{2}\left(1 - 16\tanh^2\left(\frac{z_j}{\nu_j}\right)\right)U_j + z_2\left(\upsilon_1(Z_2) - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}\right) + \sum_{j=1}^{2}\sum_{l=1}^{n}\left[(2-j+1)\frac{\rho_{jl}^2(x_l(t-\tau_l(t)))}{2} - (n-l+1)\frac{\rho_{lj}^2(x_j(t-\tau_j(t)))}{2}\right] + g_{2\mu_2}z_2z_3 \tag{31}$$

where the term $z_2(\upsilon_1(Z_2) - (\partial\alpha_1/\partial\hat{\theta})\dot{\hat{\theta}})$ will be considered later.

Remark 4: It can be known clearly from (14) that the adaptation law $\dot{\hat{\theta}}$ contains not only the current variables z_1 and z_2, but also the latter variables, namely, $z_i,\ i = 3, \dots, n$. Therefore, unlike the previous backstepping-based adaptive neural control schemes, the term $(\partial\alpha_1/\partial\hat{\theta})\dot{\hat{\theta}}$ in (26) cannot be used directly to construct the packaged uncertain functions $\hat{f}_2(Z_2)$ in (30). As a result, the function $\nu_1(Z_2)$ is introduced to compensate for $(\partial\alpha_1/\partial\hat{\theta})\dot{\hat{\theta}}$. Similarly, the function $\nu_{j-1}(Z_j)$ will be introduced to compensate for $(\partial\alpha_{j-1}/\partial\hat{\theta})\dot{\hat{\theta}}$, $j = 3, \dots, n$. The details will be shown later. The functions $\nu_{j-1}(Z_j),\ j = 2, \dots, n$, will be defined in the proof of Theorem 1.

Step k **(**$3 \le k \le n-1$**):** Similar to the design method in Step 2, choose Lyapunov function candidate $V_k = V_{k-1} + V_{z_k} + V_{P_k}$ with V_{z_k} in (11) and V_{P_k} in (12), for the following dynamic:

$$\dot{z}_k = f_k(\bar{x}_k, x_{k0}) + g_{k\mu_k}(x_{k+1} - x_{k0}) + h_k(\bar{x}_{n,\tau(t)}) - \dot{\alpha}_{k-1}$$

where

$$\dot{\alpha}_{k-1} = \sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial x_i}h_i(\bar{x}_{n,\tau(t)}) + \frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} + D_{k-1} \tag{32}$$

with $D_{k-1} = \sum_{i=1}^{k-1}(\partial\alpha_{k-1}/\partial x_i)(f_i(\bar{x}_i, x_{i0}) + g_{i\mu_i}(x_{i+1} - x_{i0})) + \omega_{k-1}$, and $\omega_{k-1} = (\partial\alpha_{k-1}/\partial\bar{y}_{d,k-1})\dot{\bar{y}}_{d,k-1} + (\partial\alpha_{k-1}/\partial\varsigma_{k-1})(\rho_{l,k-1}^2(x_{k-1}) - \rho_{l,k-1}{}^2(x_{k-1}(t-\tau))/2(1-\tau_{\max}))$.

By introducing the hyperbolic tangent function $\tanh(z_k/\nu_k)$ and a continuous function $\upsilon_{k-1}(Z_k)$, we have

$$\dot{V}_k \le \sum_{j=1}^{k}\left(-bk_{j1}z_j^2 + \frac{b\tilde{\theta}}{2\eta_j^2}S_j^T(Z_j)S_j(Z_j)z_j^2 + d_j\right) + \sum_{j=1}^{k}\left(1 - 16\tanh^2\left(\frac{z_j}{\nu_j}\right)\right)U_j + z_k g_{k\mu_k} z_{k+1} + \sum_{j=2}^{k} z_j\left(\upsilon_{j-1}(Z_j) - \frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\right) + \sum_{j=1}^{k}\sum_{l=1}^{n}\left[(k-j+1)\frac{\rho_{jl}^2(x_l(t-\tau_l(t)))}{2} - (n-l+1)\frac{\rho_{lj}^2(x_j(t-\tau_j(t)))}{2}\right] \tag{33}$$

where $d_j = \eta_j^2/2 + \varepsilon_j^2/(4bk_{j0})$ and $U_j = \sum_{l=1}^{n}((n-l+1)\rho_{lj}^2(x_j)/2(1-\tau_{\max}))$.

Step n**:** In this step, the actual control u will be constructed. For $z_n = x_n - \alpha_{n-1}$, we have

$$\dot{z}_n = f_n(\bar{x}_n, x_{n0}) + g_{n\mu_n}(u - x_{n0}) + h_n(\bar{x}_{n,\tau(t)}) - \dot{\alpha}_{n-1}$$

where $\dot{\alpha}_{n-1}$ is given in (32) with $k = n$.

Choosing V_{z_n} in (11) and using the triangular inequality, we can obtain

$$\dot{V}_{z_n} \le z_n\left(f_n(\bar{x}_n, x_{n0}) + g_{n\mu_n}(u - x_{n0}) + \frac{nz_n}{2} + \sum_{j=1}^{n-1}\frac{nz_n}{2}\left(\frac{\partial\alpha_{n-1}}{\partial x_j}\right)^2 - D_{n-1}\right) - z_n\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} + \sum_{j=1}^{n}\sum_{l=1}^{n}\frac{\rho_{jl}^2(x_l(t-\tau_l(t)))}{2}.$$

Taking Lyapunov function candidate $V_n = V_{n-1} + V_{z_n} + V_{P_n}$ with V_{n-1} in (33) and noting the derivative of V_{P_n} as

$$\dot{V}_{P_n} = \sum_{l=1}^{n}(n-l+1)\frac{\rho_{ln}^2(x_n)}{2(1-\tau_{\max})} - z_n\sum_{l=1}^{n}(n-l+1)\frac{(1-\dot{\tau}_n(t))\rho_{ln}^2(x_n(t-\tau_n(t)))}{2z_n(1-\tau_{\max})}$$

we have

$$\begin{aligned}\dot{V}_n \le &\sum_{j=1}^{n-1}\left(-bk_{j1}z_j^2 + \frac{b\tilde{\theta}}{2\eta_j^2}S_j^T(Z_j)S_j(Z_j)z_j^2 + d_j\right)\\ &+\sum_{j=1}^{n-1}\left(1-16\tanh^2\left(\frac{z_j}{\nu_j}\right)\right)U_j - z_n\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\\ &+\sum_{j=2}^{n-1}z_j\left(\upsilon_{j-1}(Z_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\right)+z_n\Bigg(g_{n\mu_n}(u-x_{n0})\\ &+\sum_{j=1}^{n-1}\frac{nz_n}{2}\left(\frac{\partial\alpha_{n-1}}{\partial x_j}\right)^2 + z_{n-1}g_{n-1\mu_{n-1}} + f_n(\bar{x}_n, x_{n0})\Bigg)\\ &+\sum_{l=1}^{n}\left((n-l+1)\frac{\rho_{ln}^2(x_n)}{2z_n(1-\tau_{\max})}+\frac{nz_n}{2}-D_{n-1}\right).\end{aligned} \tag{34}$$

Remark 5: It is clearly shown from (34) that the proposed design above has effectively eliminated the functions $h_i(\bar{x}_{n,\tau(t)})$ of delayed states, which contain multiple time-varying delays. For the sake of clarity, how to compensate for all the time-varying delay functions $h_i(\bar{x}_{n,\tau(t)})$, $i = 1,2,\ldots,n$ is summarized as follows. Firstly, in Step i, $h_i(\bar{x}_{n,\tau(t)})$ is decomposed into a series of continuous functions $\rho_{il}^2(x_l(t-\tau_l(t)))$, $l = 1,2,\ldots,n$, in terms of each delayed state based on Lemma 2. Due to the existence of unknown time-varying delays $\tau_l(t)$, the functions $\rho_{il}^2(x_l(t-\tau_l(t)))$, $l = 1,2,\ldots,n$ cannot be directly approximated by the use of RBF neural networks. Secondly, in Step i, the Lyapunov–Krasovskii functional candidate V_{Pi} in (12) is carefully designed to compensate not only for $\rho_{ii}^2(x_i(t-\tau_i(t)))$, which is obtained from the current time-delay function $h_i(\bar{x}_{n,\tau(t)})$, but also for all the time-delay terms with current delayed state $x_i(t-\tau_i(t))$, i.e., $\rho_{ji}^2(x_i(t-\tau_i(t)))$, $j = 1,2,\ldots,i-1,i+1,\ldots,n$. As a result, all the time-varying delay functions $\rho_{il}^2(x_l(t-\tau_l(t)))$ are eliminated under the sum of V_{z_i} and V_{Pi}. Last but not least, the remaining delay-free functions $\rho_{il}^2(x_l)$, which are caused by the use of Lyapunov–Krasovskii functionals to compensate for $\rho_{il}^2(x_l(t-\tau_l(t)))$, are approximated by RBF neural networks.

Next, we will make more efforts to overcome the design difficulties derived from $\sum_{l=1}^{n}(n-l+1)\rho_{ln}^2(x_n)/2z_n(1-\tau_{\max})$ and $(\partial\alpha_{n-1}/\partial\hat{\theta})\dot{\hat{\theta}}$. Similarly, employ the hyperbolic tangent function $\tanh(z_n/\nu_n)$ and a continuous function $\upsilon_{n-1}(Z_n)$ to respectively compensate for $\sum_{l=1}^{n}(n-l+1)[\rho_{ln}^2(x_n)/2z_n(1-\tau_{\max})]$ and $(\partial\alpha_{n-1}/\partial\hat{\theta})\dot{\hat{\theta}}$, and we have

$$\begin{aligned}\dot{V}_n \le &\sum_{j=1}^{n-1}\left(-bk_{j1}z_j^2 + \frac{b\tilde{\theta}}{2\eta_j^2}S_j^T(Z_j)S_j(Z_j)z_j^2 + d_j\right)\\ &+\sum_{j=1}^{n}\left(1-16\tanh^2\left(\frac{z_j}{\nu_j}\right)\right)U_j + z_n g_{n\mu_n}(u-x_{n0})\\ &+\sum_{j=2}^{n}z_j\left(\upsilon_{j-1}(Z_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\right)+z_n\hat{f}_n(Z_n)\end{aligned} \tag{35}$$

where

$$\begin{aligned}\hat{f}_n(Z_n) = &f_n(\bar{x}_n, x_{n0}) + \frac{nz_n}{2} + \sum_{j=1}^{n-1}\frac{nz_n}{2}\left(\frac{\partial\alpha_{n-1}}{\partial x_j}\right)^2 - D_{n-1}\\ &+ z_{n-1}g_{n-1\mu_{n-1}} + \frac{16\tanh^2\left(\frac{z_n}{\nu_n}\right)}{z_n}U_n - \upsilon_{n-1}(Z_n)\end{aligned}$$

with $U_n = \sum_{l=1}^{n}(n-l+1)\rho_{ln}^2(x_n)/2(1-\tau_{\max})$.

Applying the RBF neural network in (10) to approximate $\hat{f}_n(Z_n)$ on the compact set $\Omega_{Z_n}^0$, we can obtain

$$\begin{aligned}\dot{V}_n \le &\sum_{j=1}^{n-1}\left(-bk_{j1}z_j^2 + \frac{b\tilde{\theta}}{2\eta_j^2}S_j^T(Z_j)S_j(Z_j)z_j^2 + d_j\right)+d_n\\ &+\sum_{j=1}^{n}\left(1-16\tanh^2\left(\frac{z_j}{\nu_j}\right)\right)U_j + z_n g_{n\mu_n}(u-x_{n0})\\ &+\sum_{j=2}^{n}z_j\left(\upsilon_{j-1}(Z_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\right)\\ &+\frac{b\theta}{2\eta_n^2}S_n^T(Z_n)S_n(Z_n)z_n^2 + bk_{n0}z_n^2\end{aligned} \tag{36}$$

where $d_n = \eta_n^2/2+\varepsilon_n^2/(4bk_{n0})$. Constructing the actual control u in (13) and considering Remark 3, we obtain

$$g_{n\mu_n}z_n(u-x_{n0}) \le -bk_nz_n^2 - \frac{b\hat{\theta}}{2\eta_n^2}S_n^T(Z_n)S_n(Z_n)z_n^2 \tag{37}$$

with $k_n = k_{n0}+k_{n1}$. It follows from (36) and (37) that

$$\begin{aligned}\dot{V}_n \le &\sum_{j=1}^{n}\left(-bk_{j1}z_j^2 + \frac{b\tilde{\theta}}{2\eta_j^2}S_j^T(Z_j)S_j(Z_j)z_j^2 + d_j\right)\\ &+\sum_{j=1}^{n}\left(1-16\tanh^2\left(\frac{z_j}{\nu_j}\right)\right)U_j\\ &+\sum_{j=2}^{n}z_j\left(\upsilon_{j-1}(Z_j)-\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\right).\end{aligned}$$

Thus, we complete the controller design. Choosing the Lyapunov function candidate as

$$V = V_n + \frac{b\tilde{\theta}^2}{2\gamma} \tag{38}$$

we are in the position to state our main result.

Theorem 1: Consider the closed-loop system consisting of the pure-feedback nonlinear time-delay system (1) under Assumptions 1 and 2, control laws (13), and the adaptation law (14). The following properties are guaranteed under bounded initial conditions with $\hat{\theta}(t_0) \ge 0$:

1) all the signals in the closed-loop system remain uniformly ultimately bounded;
2) the vector $Z = [Z_1^T, Z_2^T, \ldots, Z_n^T]^T$ remains in a compact set $\Omega_Z^0 = \Omega_{Z_1}^0 \bigcup \Omega_{Z_2}^0, \ldots, \bigcup \Omega_{Z_n}^0$, which is specified as

$$\Omega_Z^0 = \left\{(z,\tilde{\theta},\bar{y}_{di}) \middle| |z_i| \le \mu,\ \tilde{\theta}^2 \le \frac{\gamma\mu^2}{b},\ \bar{y}_{dj}\in\Omega_{dj},\ j = 1,2,\ldots,n-1, \forall t \ge t_0\right\} \tag{39}$$

where the size of the positive constant μ depends on the initial conditions and design parameters k_j, σ, and γ;

3) the closed-loop signal $z = [z_1, z_2, \ldots, z_n] \in R^n$ will eventually converge to a compact set defined by

$$\Omega_s = \left\{ z | \quad \|z\|^2 \le 2\varrho \right\} \tag{40}$$

where $\|z\|^2 = \sum_{j=1}^{n} z_j^2$, ϱ is a constant related to the design parameters. Therefore, Ω_s can be made as small as desired using a trial-and-error method to obtain the appropriate design parameters.

Proof: It follows from Lemma 3 and the definition of U_j in (33) for $z_j \in \Omega^0_{Z_j}$, $j = 1, \ldots, n$, that

$$\sum_{j=1}^{n} \left(1 - 16\tanh^2\left(\frac{z_j}{\nu_j}\right)\right) U_j \le 0.$$

Then, (36) can be rewritten as

$$\dot{V}_n \le \sum_{j=1}^{n} \left(-bk_{j1}z_j^2 + \frac{b\tilde{\theta}}{2\eta_j^2} S_j^T(Z_j)S_j(Z_j)z_j^2 + d_j \right) + \sum_{j=2}^{n} z_j \left(\upsilon_{j-1}(Z_j) - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} \right).$$

Considering the Lyapunov function candidate V in (38), its time derivative is

$$\dot{V} \le \sum_{j=1}^{n} \left(-bk_{j1}z_j^2 + d_j) + \sum_{j=2}^{n} z_j (\upsilon_{j-1}(Z_j) - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} \right) + \frac{b\tilde{\theta}}{\gamma} \left(\sum_{j=1}^{n} \frac{\gamma}{2\eta_j^2} S_j^T(Z_j)S_j(Z_j)z_j^2 - \dot{\hat{\theta}} \right). \tag{41}$$

In view of the adaptation law $\hat{\theta}$ in (14), (41) is further written as

$$\dot{V} \le \sum_{j=1}^{n} \left(-bk_{j1}z_j^2 + d_j \right) + \frac{b\sigma\hat{\theta}\tilde{\theta}}{\gamma} + \sum_{j=2}^{n} z_j \left(\upsilon_{j-1}(Z_j) - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} \right). \tag{42}$$

From (42), the key problem of the proof is how to appropriately choose the smooth function $\upsilon_{j-1}(Z_j)$ to eliminate the last term in (42). Next, we will determine $\upsilon_{j-1}(Z_j)$ such that $\sum_{j=2}^{n} z_j(\upsilon_{j-1}(Z_j) - (\partial\alpha_{j-1}/\partial\hat{\theta})\dot{\hat{\theta}}) \le 0$. By the definition of the adaptation law $\dot{\hat{\theta}}$ in (14) and noting Lemma 1, we have

$$\begin{aligned} &-\sum_{j=2}^{n} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} \\ &= -\sum_{j=2}^{n} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \left(\sum_{i=1}^{n} \frac{\gamma S_i^T(Z_i)S_i(Z_i)z_i^2}{2\eta_i^2} - \sigma\hat{\theta} \right) \\ &\le \sum_{j=2}^{n} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\sigma\hat{\theta} + \sum_{j=2}^{n}\sum_{i=j}^{n} \frac{\gamma s^2 z_i^2}{2\eta_i^2} \left| z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \right| \\ &\quad - \sum_{j=2}^{n} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \sum_{i=1}^{j-1} \frac{\gamma S_i^T(Z_i)S_i(Z_i)z_i^2}{2\eta_i^2} \\ &= \sum_{j=2}^{n} z_j \left(\frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\sigma\hat{\theta} - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \sum_{i=1}^{j-1} \frac{\gamma S_i^T(Z_i)S_i(Z_i)z_i^2}{2\eta_i^2} + \sum_{i=2}^{j} \frac{\gamma s^2 z_j}{2\eta_j^2} \left| z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right| \right). \end{aligned}$$

Then, choosing $\upsilon_{j-1}(Z_j)$ as

$$\upsilon_{j-1}(Z_j) = \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \sum_{i=1}^{j-1} \frac{\gamma S_i^T(Z_i)S_i(Z_i)z_i^2}{2\eta_i^2} - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\sigma\hat{\theta} - \sum_{i=2}^{j} \frac{\gamma s^2 z_j}{2\eta_j^2} \left| z_i \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right|$$

we have

$$\sum_{j=2}^{n} z_j \left(\upsilon_{j-1}(Z_j) - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} \right) \le 0. \tag{43}$$

Substituting (43) into (42) and noting $\hat{\theta} = \theta - \tilde{\theta}$, we obtain

$$\dot{V} \le \sum_{j=1}^{n} \left(-bk_{j1}z_j^2 + d_j \right) + \frac{b\sigma\tilde{\theta}\theta}{\gamma} \le -\left(\sum_{j=1}^{n} bk_{j1}z_j^2 + \frac{b\sigma\tilde{\theta}^2}{2\gamma} \right) + C \tag{44}$$

where $C = \sum_{j=1}^{n} d_j + b\sigma\theta^2/(2\gamma)$ with $d_j = \eta_j^2/2 + \varepsilon_j^2/(4bk_{j0})$.

It follows from the definition of control gains k_i in (13) that

$$\dot{V} \le -\sum_{j=1}^{n} bk_{j2}z_j^2 - \frac{b\sigma\tilde{\theta}^2}{2\gamma} - \sum_{j=1}^{n} \frac{b\epsilon_j \cosh(z_j)z_j^2}{1+z_j^2} \sum_{l=1}^{n} (n-l+1) \int_{t-\tau}^{t} \frac{\rho_{lj}^2(x_j(\sigma))}{2(1-\tau_{\max})} d\sigma + C. \tag{45}$$

Since $[t - \tau_j(t), t] \subset [t - \tau, t]$ based on $\tau_j(t) \le \tau$, we have $\int_{t-\tau_j(t)}^{t} (\rho_{lj}^2(x_j(\sigma))/2(1-\tau_{\max}))d\sigma \le \int_{t-\tau}^{t} (\rho_{lj}^2(x_j(\sigma))/2(1-\tau_{\max}))d\sigma$. Moveover, b and ϵ_j are positive definite, it can be obtained in terms of (9) that $\psi(z_j) = \cosh(z_j)z_j^2/(1+z_j^2) > 0$. Then, (45) becomes

$$\dot{V} \le -\sum_{j=1}^{n} bk_{j2}z_j^2 - \sum_{j=1}^{n} b\epsilon_j \psi(z_j) \sum_{l=1}^{n} (n-l+1) \int_{t-\tau_i(t)}^{t} \frac{\rho_{lj}^2(x_j(\sigma))}{2(1-\tau_{\max})} d\sigma - \frac{b\sigma\tilde{\theta}^2}{2\gamma} + C. \tag{46}$$

Based on the construction of the quadratic function V_{z_i} in (11) and the Lyapunov–Krasovskii functional candidate V_{P_i} in (12), and the property of even function $\psi(z_j)$ in (9), we can obtain from (38)

$$\dot{V} \le -aV + C \tag{47}$$

which implies that

$$V \le (V(t_0) - \varrho)e^{-a(t-t_0)} + \varrho \le V(t_0) + \varrho, \quad \forall t \ge t_0 \tag{48}$$

where $a = \min\{2bk_{j2}, b\epsilon_j \psi(0.2554\nu_j), \sigma\}$, $\varrho = C/a$. From (48), it is known that V, z_j, and $\tilde{\theta}$, $i = 1, 2, \ldots, n$, are bounded. $\hat{\theta} = \theta - \tilde{\theta}$ is bounded because of the boundedness of θ and $\tilde{\theta}$. For $z_1 = x_1 - y_d$ and y_d being bounded, x_1 is bounded. Since α_1 is a function of bounded signals x_1, y_d, $\dot{y}_d$, and $\hat{\theta}$, α_1 is also bounded. From $x_2 = z_2 + \alpha_1$, x_2 is bounded. Similarly, α_{i-1} and x_i, $i = 3, \ldots, n$, are bounded. Therefore, all the signals of the closed-loop system are bounded.

Considering the definition of V in (38) and applying (48), the following inequalities hold:

$$\sum_{j=1}^{n} \frac{z_j^2}{2} \le V \le V(t_0) + \varrho, \qquad \frac{b\tilde{\theta}^2}{2\gamma} \le V \le V(t_0) + \varrho. \tag{49}$$

Let $\mu = \sqrt{2(V(t_0) + \varrho)}$, we have

$$|z_j| \le \mu, \qquad \tilde{\theta}^2 \le \frac{\gamma \mu^2}{b}. \tag{50}$$

Therefore, we have the compact set $\Omega^0_{Z_j}$ in Theorem 1 over which the RBF neural universal approximation is applied with its feasibility.

In addition, according to (11) and (38), we have that $\sum_{j=1}^{n} z_j^2/2 \le V$. Using the first inequality in (48), the following inequality holds:

$$\lim_{t\to\infty} \|z\| \le \sqrt{2\varrho}.$$

Note that k_{j0}, k_{j2}, ϵ_j, η_j, ν_j, σ_j, and γ_j, $i = 1, 2, \ldots, n$, are design parameters, b, ε_i and θ are constants. Therefore, by appropriately online-tuning the design parameters, the compact set Ω_s can be made as small as desired. Subsequently, the tracking error $z_1 = y - y_d$ can be shown to converge to a small neighborhood of the origin. This completes the proof.

Remark 6: It should be pointed out that the proposed control algorithm for pure-feedback nonlinear systems with unknown state time-varying delays is very different from the previous work on the pure-feedback delay-free system. The main differences are as follows:

1) from the system (1), every time-delay function $h_i(\bar{x}_{n,\tau(t)})$ contains not only time-varying delay states $x_1(t - \tau_1(t)), \ldots, x_i(t - \tau_i(t))$ of the previous channels, but also all the later delay states $x_{i+1}(t - \tau_{i+1}(t)), \ldots, x_n(t - \tau_n(t))$. The time-delay functions $h_i(\bar{x}_{n,\tau(t)})$, $i = 1, 2, \ldots, n$, maybe cause the circular construction of controller and the singularity problem. Therefore, to reduce the design difficulty caused by the time-varying delay functions $h_i(\bar{x}_{n,\tau(t)})$, the separation technique in Lemma 2 is used to decompose $h_i(\bar{x}_{n,\tau(t)})$ into a series of continuous functions $\rho_{il}^2(x_l(t - \tau_l(t)))$, $l = 1, 2, \ldots, n$, of each delayed state $x_l(t - \tau_l(t))$. This method avoids the circular construction of controller and the use of any assumption on unknown time-delay functions $h_i(\bar{x}_{n,\tau(t)})$;
2) to compensate for the time-delay functions $\rho_{1l}^2(x_l(t - \tau_l(t)))$ obtained by decomposing $h_1(\bar{x}_{n,\tau(t)})$, we recursively construct the novel Lyapunov–Krasovskii functionals V_{Pl} in (12), which can completely compensate for all the time-delay functions until the last step of backstepping in (34). However, the use of Lyapunov–Krasovskii functionals V_{Pi} in (12) will induce the remaining term $\sum_{l=1}^{n}((n - l + 1)\rho_{lj}^2(x_j)/2z_j(1 - \tau_{\max}))$ in (20), (28) and (34). The term makes it be infeasible to be approximated by an RBF neural network, since it is discontinuous at $z_j = 0$. Therefore, hyperbolic tangent function $\tanh(z_j/\nu_j)$ has to be introduced in (21), (29), and (35) to overcome the singularity problem;
3) although the circular construction of controller and the singularity problem caused by time-delay functions $h_i(\bar{x}_{n,\tau(t)})$ have been solved, it is very difficult from (44) to deduce the stability result $\dot{V} = -aV + C$ with a and C being constants. The main reason comes from the use of Lyapunov–Krasovskii functionals V_{Pi} in the global Lyapunov function V. Therefore, we must put our efforts to make the proof be carried over. Based on the form of V_{Pi} in (12), the control gains k_{j1} are modified as a dynamic form with a even function, i.e., $k_{i1} = k_{i2} + (\epsilon_i \cosh(z_i)/1 + z_i^2) \sum_{l=1}^{n}(n - l + 1) \int_{t-\tau}^{t} (\rho_{li}^2(x_i(\sigma))/2(1 - \tau_{\max}))d\sigma$. The modified control gains k_{j1} make the stability analysis to be carried out. [See the inequalities (45) and (46) for details.]

IV. Simulation Studies

Example 1: To illustrate the validity of the proposed adaptive neural control, a class of pure-feedback nonlinear systems with multiple time-varying delay states is simulated, which is described by the following differential equation:

$$\begin{cases} \dot{x}_1 = \frac{1-e^{-x_1}}{1+e^{-x_1}} + x_2^3 + x_2 e^{-1-x_1^2} + h_1(\bar{x}_{2,\tau(t)}) \\ \dot{x}_2 = x_1^2 + 0.1(1 + x_2^2)u + f(u) + h_2(\bar{x}_{2,\tau(t)}) \\ y = x_1 \end{cases} \tag{51}$$

where $f(u) = 0.15u^3 + \sin(0.1u)$, $h_1(\bar{x}_{2,\tau(t)}) = 0.5x_1^2(t - \tau_1(t))x_2(t - \tau_2(t))$, $h_2(\bar{x}_{2,\tau(t)}) = x_1(t - \tau_1(t))x_2(t - \tau_2(t))$.

Choose the reference signal as $y_d = \sin(0.5t) + 0.5\sin(1.5t)$. The control objective is to design an adaptive neural tracking control for System (51) such that all the signals in the closed-loop system remain bounded and the system output y follows the given reference signal y_d.

Based on Theorem 1, the adaptive neural control law is chosen as

$$u = -k_2 z_2 - \frac{\hat{\theta}}{2\eta_2^2} S_2^T(Z_2)S_2(Z_2)z_2 + x_{20} \tag{52}$$

where $Z_2 = [x_1, x_2, \hat{\theta}, \varsigma_1, \omega_1]^T$ with $\varsigma_1 = \int_{t-\tau}^{t}(\rho_{l1}^2(x_1(\sigma))/2(1 - \tau_{\max}))d\sigma$, $\omega_1 = (\partial\alpha_1/\partial\bar{y}_{d1})\dot{\bar{y}}_{d1} + (\partial\alpha_1/\partial\varsigma_1)(\rho_{l1}^2(x_1) - \rho_{l1}^2(x_1(t - \tau))/\ 2(1 - \tau_{\max}))$, and choose the virtual control law and adaptation law as

$$\alpha_1 = -k_1 z_1 - \frac{\hat{\theta}}{2\eta_1^2} S_1^T(Z_1)S_1(Z_1)z_1 + x_{10},$$

$$\dot{\hat{\theta}} = \sum_{i=1}^{2} \frac{\gamma}{2\eta_i^2} S_i^T(Z_i)S_i(Z_i)z_i^2 - \sigma\hat{\theta}.$$

In the simulation, choose initial conditions $[x_1(t), x_2(t)]^T = [0.2, 0.5]^T$, time-varying delays $\tau_1(t) = 0.2(1 + \sin(t))$,

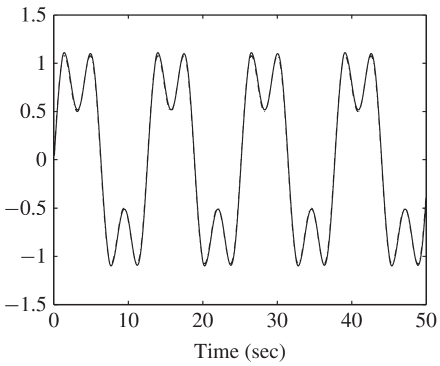

Fig. 1. System output y (–) and reference signal y_d (-.-) of Example 1.

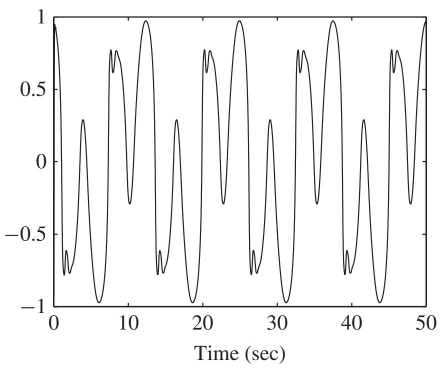

Fig. 2. State variable x_2 of Example 1.

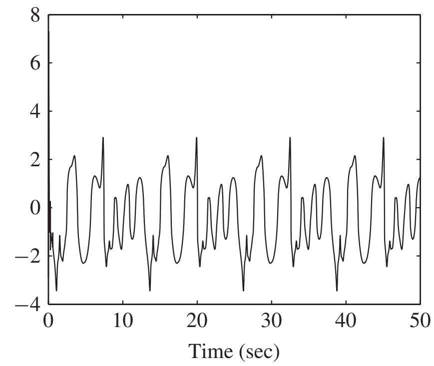

Fig. 3. Control u of Example 1.

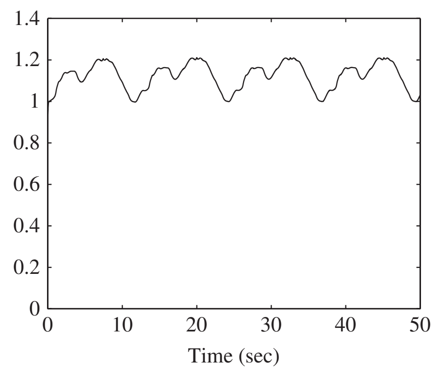

Fig. 4. Adaptive parameter $\hat{\theta}$ of Example 1.

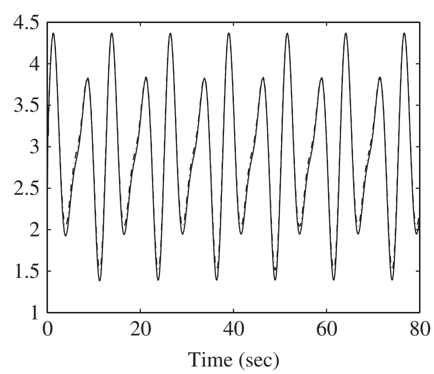

Fig. 5. System output y (–) and reference signal y_d (-.-) of Example 2.

$\tau_2(t) = 1 - 0.5\cos(t)$, and $\hat{\theta}(0) = 0$. Some other parameters are chosen as follows: $k_1 = 5$, $k_2 = 5$, $\epsilon_1 = 1$, $\epsilon_2 = 0.5$, $\eta_1 = \eta_2 = 0.25$, $\gamma = 5$, $\sigma = 0.1$, $x_{10} = 0.2$, and $x_{20} = 0.5$. Moreover, in the simulation we choose the RBF neural networks in the following way. Neural network $W_1^T S_1(Z_1)$ contains 75 nodes with centers spaced evenly in the interval $[-1.5, 1.5] \times [-1.5, 1.5] \times [-1.5, 1.5]$ and widths equal to 2. Neural network $W_2^T S_2(Z_2)$ contains 1125 nodes with centers spaced evenly in the interval $[-1.5, 1.5] \times [-1.5, 1.5] \times [0, 2] \times [-3, 3] \times [-2, 2]$ and widths equal to 2.

Simulation results are shown in Figs. 1–4, respectively. Fig. 1 shows the system output y and the reference signal y_d. From Fig. 1, it can be seen that the tracking performance has been achieved. Fig. 2 shows the response of state variable x_2, Fig. 3 displays the control input signal u, and Fig. 4 shows the response curve of adaptive parameter $\hat{\theta}$. Obviously, simulation results show that the controller works well.

Example 2: To show the applicability of our result, consider the following Brusselator model in dimensionless form, which comes from [42]:

$$\begin{cases} \dot{x}_1 = C - (D+1)x_1 + x_1^2 x_2 + d_1(t,x) \\ \dot{x}_2 = Dx_1 - x_1^2 x_2 + (2+\cos(x_1))u + d_2(t,x) \\ y = x_1 \end{cases} \tag{53}$$

where x_1 and x_2 denote the concentrations of the reaction intermediates, C, $D > 0$ are parameters which describe the supply of "reservoir" chemicals. $d_1(t,x)$ and $d_2(t,x)$ are the external disturbance terms. The model was introduced in detail in [43] and [44]. In this paper, the external disturbance terms $d_1(t,x)$ and $d_2(t,x)$ are ignored. To study the effect caused by time-varying delays, we add time-delay terms $h_i(\bar{x}_{2,\tau(t)})$ in (53) and obtain the following time-delay Brusselator model:

$$\begin{cases} \dot{x}_1 = C - (D+1)x_1 + x_1^2 x_2 + h_1(\bar{x}_{2,\tau(t)}) \\ \dot{x}_2 = Dx_1 - x_1^2 x_2 + (2+\cos(x_1))u + h_2(\bar{x}_{2,\tau(t)}) \\ y = x_1 \end{cases}$$

where $h_1(\bar{x}_{2,\tau(t)})$ and $h_2(\bar{x}_{2,\tau(t)})$ are unknown delay-state terms, which are chosen as the functions $2\cos(x_1(t - \tau_1(t)))x_2(t-\tau_2(t))$, $0.2\sin(x_2(t-\tau_2(t)))x_1(t-\tau_1(t))$, respectively. Obviously, the Brusselator model is in the cascade form (1) under the assumption that $x_1 \neq 0$ [42]. In the simulation, we choose the reference signal $y_d = 3 + \sin(t) + 0.5\sin(1.5t)$, $C = 1$, $D = 3$, $\tau_1(t) = -0.5\cos(t)$, $\tau_2 = 1 + 0.5\sin(t)$, the upper bound of these time-varying delays and their derivatives is $\tau = 1.5$ and $\tau_{\max} = 0.5$, respectively, RBF neural network $W_1^T S_1(Z_1)$ are chosen by containing 81 nodes with centers

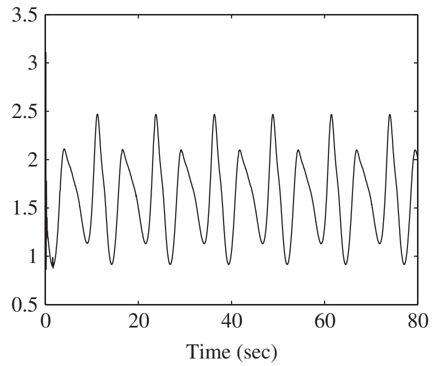

Fig. 6. State variable x_2 of Example 2.

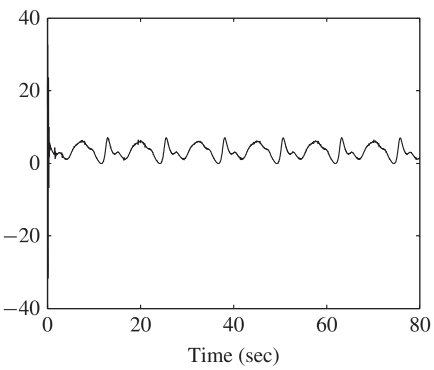

Fig. 7. Control u of Example 2.

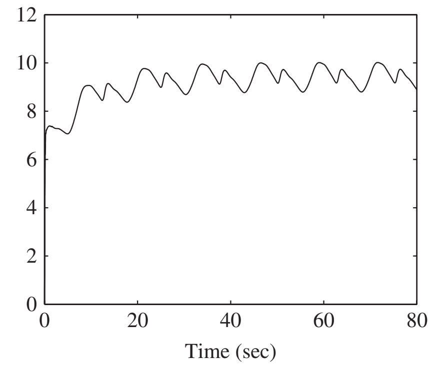

Fig. 8. Adaptive parameter $\hat{\theta}$ of Example 2.

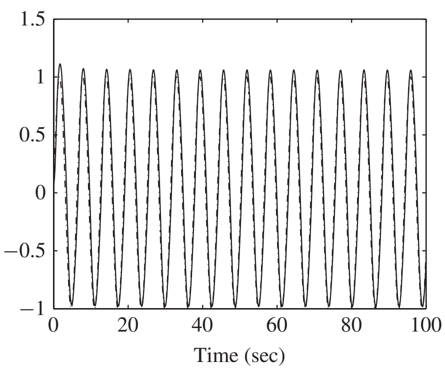

Fig. 9. System output y (–) and reference signal y_d (-.-) of Example 3.

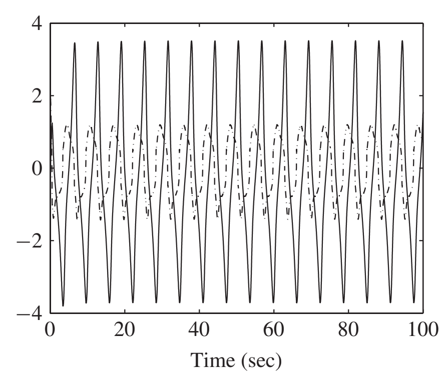

Fig. 10. State variable x_2 of Example 3.

spaced evenly in the interval $[1,4] \times [1,4] \times [-1.5, 1.5]$ and widths being equal to 2, neural network $W_2^T S_2(Z_2)$ is constructed by containing 1125 nodes with centers spaced evenly in the interval $[1,4] \times [1,4] \times [0,6] \times [-3,3] \times [-2,2]$ and widths equal to 2.

In the simulation, we choose design parameters of the controller (52) as follows: $k_{10} = k_{11} = 3$, $k_{20} = k_{21} = 4$, $\epsilon_1 = \epsilon_2 = 0.5$, $\eta_1 = 0.25$, $\eta_2 = 0.35$, $\gamma = 6$, $\sigma = 0.05$, and $x_{10} = x_{20} = 0$. The simulation is run under the initial conditions $[x_1(t), x_2(t)]^T = [2.5, 1]^T$ for $-\tau \le t \le 0$, and $\theta(0) = 0$. Simulation results are shown in Figs. 5–8, respectively. Fig. 5 shows that the system output y can follow the given reference signal y_d. Figs. 6–8 show the responses of other variables x_2, u, $\hat{\theta}$. Obviously, simulation results show that the controller works well and achieves the desired convergence performance.

Example 3: To further show the control capability of the proposed control scheme, we consider the following third-order nonlinear time-delay system:

$$\begin{cases} \dot{x}_1 = x_1 \sin(x_1) + (0.5 + x_1^2)x_2 + x_1(t - \tau_1(t)) \\ \dot{x}_2 = x_2 e^{-0.5x_1} + (1 + x_2^2)x_3^3 + h_2(\bar{x}_{3,\tau(t)}) \\ \dot{x}_3 = x_1 x_2 x_3 + f(x_1, x_2, u) + h_3(\bar{x}_{3,\tau(t)}) \\ y = x_1 \end{cases} \tag{54}$$

where $h_2(\bar{x}_{3,\tau(t)}) = x_1(t - \tau_1(t))x_2(t - \tau_2(t))x_3(t - \tau_3(t))$, $h_3(\bar{x}_{3,\tau(t)}) = x_2(t - \tau_2(t))x_3(t - \tau_3(t))$, and $f(x_1, x_2, u) = \cos(u) + (2 + \cos(x_1 x_2))u$.

It can be clearly seen that the system (54) is consistent with the structure of the studied system (1). By Theorem 1, the virtual control laws α_i, the true control law u, and the adaptive laws $\dot{\hat{\theta}}$ are chosen respectively as

$$\alpha_i = -k_i z_i - \frac{\hat{\theta}}{2\eta_i^2} S_i^T(Z_i) S_i(Z_i) z_i + x_{i0}$$

$$\dot{\hat{\theta}} = \sum_{i=1}^{3} \frac{\gamma}{2\eta_i^2} S_i^T(Z_i) S_i(Z_i) z_i^2 - \sigma\hat{\theta}$$

where $i = 1, 2, 3$, $z_1 = x_1 - y_d$ with the reference signal $y_d = \sin(t)$, $z_2 = x_2 - \alpha_1$, $z_3 = x_3 - \alpha_2$, $Z_1 = [x_1, y_d, \dot{y}_d]^T$, $Z_2 = [x_1, x_2, \hat{\theta}, \varsigma_1, \omega_1]^T$, and $Z_3 = [x_1, x_2, x_3, \hat{\theta}, \varsigma_2, \omega_2]^T$. In the simulation, the design parameters are chosen as follows: $k_1 = 4$, $k_2 = 8$, $k_3 = 8$, $\epsilon_1 = 0.5$, $\epsilon_2 = \epsilon_3 = 1$, $\eta_1 = 0.5$, $\eta_2 = \eta_3 = 0.6$, $\gamma = 0.35$, and $\sigma = 0.03$. Moreover, the same RBF neural networks $W_1^T S_1(Z_1)$ and $W_2^T S_2(Z_2)$ are taken to be the same as in Example 1, and $W_3^T S_3(Z_3)$ is chosen to contain 729 nodes with centers spaced evenly in the interval

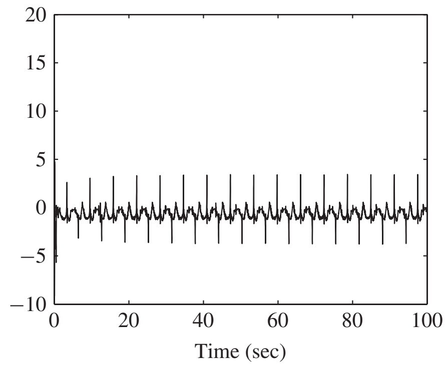

Fig. 11. Control u of Example 3.

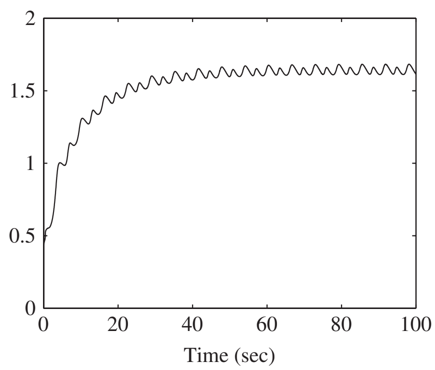

Fig. 12. Adaptive parameter $\hat{\theta}$ of Example 3.

$[-3,3] \times [-3,3] \times [-3,3] \times [0,3] \times [0,3] \times [-1,1]$ and widths being equal to 4.

In the simulation, we assume time delays $\tau_1(t) = 0.2\sin(t)$, $\tau_2(t) = 0.5\cos(t)$, and $\tau_3 = 0.5 + 0.5\sin(t)$, then $\tau = 1$ and $\tau_{\max} = 0.5$. Under initial conditions $[x_1(t), x_2(t), x_3(t)]^T = [0.1, -0.5, 0.1]^T$ for $-\tau \le t \le 0$, and $\hat{\theta}(0) = 0$, simulation is run under $x_{10} = x_{20} = x_{30} = 0$, and the results show that under the action of the presented controller, good convergence performance is achieved for the system (54). The details are shown in Figs. 9–12.

Remark 7: Usually, the tracking performance depends on the design parameters of adaptive neural controller (13). Theoretically, a good tracking performance can be achieved by choosing control gains k_i sufficiently large or η_i sufficiently small. However, how to choose the optimal parameters, such as k_i, η_i, σ, and so on, to achieve the optimal tracking performance is still an open problem. In the presented simulations, the design parameters are set using a trial-and-error method.

On the other hand, the proposed scheme is especially suitable to the control of higher order nonlinear systems since only one parameter $\hat{\theta}$ in (14) needs to be online-tuned. As such, the computational speed of the scheme is improved greatly (see Figs. 1, 5, 9 for details). As mentioned in [45], as the system order increases, there will be more inputs n to the neural networks, which will cause the network to contain at least 5^n nodes by supposing at least five evenly spaced centers for each input. Consequently, it can be clearly seen that there are a large number of parameters needed to be online-tuned because of the weight values W_i themselves being used as the estimated parameters in the previous adaptive neural control schemes.

V. Conclusion

A simple and effective control approach has been presented for non-affine pure-feedback system with multiple time-varying delay states. The use of separation technique and the norm of neural weight vector not only avoids any restriction on unknown time-delay terms with all time-varying delay states, but also overcomes the curse of dimensionality of adaptive parameters. The proposed control scheme has been proven to be able to guarantee the boundedness of all the closed-loop signals. Simulation results have illustrated the effectiveness of the proposed scheme.

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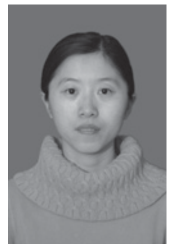

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